Propagation of partially coherent pulsed beams in the spatiotemporal domain

Li-gang Wang,[1] Qiang Lin,[2] Hong Chen,[3] and Shi-yao Zhu[1]
[1]*Department of Physics, Hong Kong Baptist University, Kowloon Tong, Hong Kong, China*
[2]*Institute of Optics, Department of Physics, Zhejiang University, Hangzhou 310028, China*
[3]*Department of Physics, Tongji University, Shanghai 200092, China*
(Received 9 September 2002; revised manuscript received 30 January 2003; published 19 May 2003)

A generalized model to describe the spatiotemporal partially coherent pulsed beams is presented. The corresponding propagation formula is derived by using the partially coherent light theory. Based on this formula, we obtain a nonstationary generalized *ABCD* law (which illustrates the transformation of optical beams or pulses passing through media) to describe the spatiotemporal behavior of partially coherent Gaussian pulsed beams. The physical meaning of such generalized pulsed beams is discussed. An example to illustrate the application of this law is given.

DOI: 10.1103/PhysRevE.67.056613 PACS number(s): 42.25.Kb, 41.85.Ew, 42.65.Re

I. INTRODUCTION

A partially coherent field and its fluctuations have been extensively investigated. In the previous studies, the partially coherent light field is mainly stationary and ergodic [1–3], i.e., its ensemble averages (the statistical averages) are essentially stationary. For a nonstationary light field, such as pulses, they have been traditionally treated as completely spatiaotemporally coherent light. For nonstationary light fields, the average values, such as correlation functions $\Gamma(\vec{r}_1,t_1;\vec{r}_2,t_2)$, are dependent on the choice of the original point of time, and hence depend on two time variables. In such cases, we have to use the generalized Wiener-Khintchine theorem, which involves two time variables, for gaining a knowledge of the spectrum of the nonstationary light fields [4,5]. Recently, incoherent spatial solitons and incoherent light pulses have attracted a lot of attentions in experimental and theoretical studies [6–8]. For partially coherent pulses, the random method is often used [8], where the phase or amplitude is fluctuated within a range. Another way to study the coherence properties of nonstationary light fields is the spectral approach [5,9–11], which is a generalization of the coherence theory [1,2] from a stationary case into a nonstationary case in space-frequency domain. Based on the theory of coherence for a nonstationary light field, Pääkkönen *et al.* discussed the partially coherent Gaussian light pulses [12]. Such light pulses are composed of partially correlated frequency components in frequency domain, and in a time domain they are partially coherent. In this paper, we will investigate the partially coherent Gaussian pulsed beam by directly considering the correlation function in the spatiotemporal domain, not by the spectral correlation function in space-frequency domain. Our method is more convenient to analyze the spatiotemporal behaviors of such pulsed beams than those of the spectral approach.

For any practical pulsed beams, they are not only temporally confined, but also transversely spatially confined. In this paper, from second-order dispersive equations for pulsed beams [13], we can obtain two differential equations for the correlation functions of any partially coherent pulsed beams and their solutions in a spatiotemporal domain. These solutions describe the propagation of a partially coherent pulsed beam through the second-order dispersive media. Then we present a more general model to characterize partially coherent pulsed beams in a spatiotemporal domain which may be called Gaussian Schell-model pulsed beams (GSMPBs). This model helps us to understand the effects of the transverse spatial coherence and temporal coherence on the evolutions of the pulsed beams. We assume that both the spatial and temporal intensity profiles and the transverse spatial and temporal coherences of pulses are in Gaussian functions. Furthermore, we get an analytical expression of the correlation function of spatiotemporally coupled GSMPBs through the second-order dispersive media in tensor form, and we call it the generalized *ABCD* law for partially coherent Gaussian Schell-model pulsed beams. These results are useful in a wide range of studies on pulses and their propagations, such as characteristics of ultrashort pulses, pulse compression, the interaction between pulses with matter, etc. This model also helps us to get deeper insights into the spatiotemporal behavior of the pulses. An example to illustrate the application of this generalized *ABCD* law is given, which shows the effects of the transverse spatial coherence on the spatial-temporal behavior of the pulsed beam.

II. PROPAGATION FORMULA OF CORRELATION FUNCTION OF PARTIALLY COHERENT PULSED BEAMS IN DISPERSIVE MEDIA

First, we consider the spatial-temporal propagation properties of partially coherent pulsed beam. Let $E(x,y,z,t)$ represents the fluctuated electric field at time t and position (x,y,z). It has a well-defined central frequency ω_0 and the corresponding wave number is $k_0=\omega_0/c$:

$$E(x,y,z,t)=V(x,y,z,t)e^{i(\omega_0 t-k_0 z)}, \tag{1}$$

where $V(x,y,z,t)$ is the paraxial slowly varying envelope of the field, which is generally a complex function depending on time t and potion (x,y,z). In a linear dispersive medium, when we take into account up to the second-order dispersion, $V(x,y,z,t)$ satisfies the following equation [13]:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]V(x,y,z,t)+2ik_0\left[\frac{\partial}{\partial z}+\beta'\frac{\partial}{\partial t}\right]V(x,y,z,t)+k_0\beta''\frac{\partial^2}{\partial t^2}V(x,y,z,t)=0, \tag{2}$$

where β' is the inverse group velocity, and β'' is the group velocity dispersion. By taking the variable transforms

$$\xi=z,$$

$$\tau=\frac{1}{[k_0\beta'']^{1/2}}(t-\beta' z), \tag{3}$$

and then substituting Eq. (3) into Eq. (2), we get

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial \tau^2}\right]V(x,y,\xi,\tau)+2ik_0\frac{\partial}{\partial \xi}V(x,y,\xi,\tau)=0. \tag{4}$$

Taking the complex conjugate of Eq. (4), and writing x_1, y_1, and τ_1 in place of x, y, and τ, respectively, we obtain the equation

$$\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial \tau_1^2}\right]V^*(x_1,y_1,\xi_1,\tau_1)-2ik_0\frac{\partial}{\partial \xi_1}V^*(x_1,y_1,\xi_1,\tau_1)=0. \tag{5}$$

Next we multiply both sides of Eq. (5) by $V(x_2,y_2,\xi_2,\tau_2)$, and take the ensemble average of Eq. (5) over the different realizations of the field. By interchanging the order of the ensemble average and the differential operators, we obtain the equation

$$\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial \tau_1^2}\right]\Gamma(x_1,y_1,\xi_1,\tau_1;x_2,y_2,\xi_2,\tau_2)-2ik_0\frac{\partial}{\partial \xi_1}\Gamma(x_1,y_1,\xi_1,\tau_1;x_2,y_2,\xi_2,\tau_2)=0, \tag{6}$$

where $\Gamma(x_1,y_1,\xi_1,\tau_1;x_2,y_2,\xi_2,\tau_2)$ is the second-order correlation function defined by

$$\Gamma(x_1,y_1,\xi_1,\tau_1;x_2,y_2,\xi_2,\tau_2)=\langle V_1^*(x_1,y_1,\xi_1,\tau_1)V_2(x_2,y_2,\xi_2,\tau_2)\rangle, \tag{7}$$

where $\langle\,\rangle$ represents the ensemble average. In a similar way, we can derive the equation

$$\left[\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial y_2^2}+\frac{\partial^2}{\partial \tau_2^2}\right]\Gamma(x_1,y_1,\xi_1,\tau_1;x_2,y_2,\xi_2,\tau_2)+2ik_0\frac{\partial}{\partial \xi_2}\Gamma(x_1,y_1,\xi_1,\tau_1;x_2,y_2,\xi_2,\tau_2)=0. \tag{8}$$

Equations (6) and (8) describe the propagations of any partially coherent pulsed beams through the second-order dispersive media. It should be pointed out that both Eqs. (6) and (8) are more general than the classic Wolf's equations [2] that are valid for statistically stationary light fields, while the present equations are valid for nonstationary fields. At the same time, however, these two equations are more restrictive than Wolf's equations due to the paraxial approximation in our derivation. The integral solution of Eqs. (6) and (8) reads

$$\Gamma(x_1,y_1,\xi_1,\tau_1;x_2,y_2,\xi_2,\tau_2)=\left(\frac{k_0}{2\pi}\right)^3\left(\frac{1}{\det(\tilde{\mathbf{B}}_1\tilde{\mathbf{B}}_2)}\right)^{1/2}\int\int\int\int\int\int\Gamma_0(x_{10},y_{10},\xi_{10},\tau_{10};x_{20},y_{20},\xi_{20},\tau_{20})$$
$$\times\exp\left\{-\frac{ik_0}{2}[(\tilde{\mathbf{r}}_{10}^T\tilde{\mathbf{B}}_1^{-1}\tilde{\mathbf{A}}_1\tilde{\mathbf{r}}_{10}+\tilde{\mathbf{r}}_1^T\tilde{\mathbf{D}}_1\tilde{\mathbf{B}}_1^{-1}\tilde{\mathbf{r}}_1-2\tilde{\mathbf{r}}_{10}^T\tilde{\mathbf{B}}_1^{-1}\tilde{\mathbf{r}}_1)-(\tilde{\mathbf{r}}_{20}^T\tilde{\mathbf{B}}_2^{-1}\tilde{\mathbf{A}}_2\tilde{\mathbf{r}}_{20}+\tilde{\mathbf{r}}_2^T\tilde{\mathbf{D}}_2\tilde{\mathbf{B}}_2^{-1}\tilde{\mathbf{r}}_2-2\tilde{\mathbf{r}}_{20}^T\tilde{\mathbf{B}}_2^{-1}\tilde{\mathbf{r}}_2)]\right\}d^3\tilde{\mathbf{r}}_{10}d^3\tilde{\mathbf{r}}_{20}, \tag{9}$$

where $\tilde{\mathbf{A}}_1,\tilde{\mathbf{B}}_1,\tilde{\mathbf{D}}_1$ and $\tilde{\mathbf{A}}_2,\tilde{\mathbf{B}}_2,\tilde{\mathbf{D}}_2$ are the 3×3 spatiotemporal **ABCD** matrices of the dispersive optical elements [14,15], and $\Gamma_0(x_{10},y_{10},\xi_{10},\tau_{10};x_{20},y_{20},\xi_{20},\tau_{20})$ is the initial correlation function of pulsed beams. Equation (9) can be rewritten in a more compact form in terms of tensors as follows:

$$\Gamma(\tilde{\mathbf{r}}_1;\tilde{\mathbf{r}}_2)=\left(\frac{k_0}{2\pi}\right)^3[\det((-1)^m\overline{\mathbf{B}})]^{-1/2}\int\int\int\int\int\int\Gamma_0(\tilde{\mathbf{r}}_{10};\tilde{\mathbf{r}}_{20})$$
$$\times\exp\left[-i\frac{k_0}{2}\begin{pmatrix}\overline{\mathbf{r}}_0\\\overline{\mathbf{r}}\end{pmatrix}^T\begin{pmatrix}\overline{\mathbf{B}}^{-1}\overline{\mathbf{A}} & -\overline{\mathbf{B}}^{-1}\\ \overline{\mathbf{C}}-\overline{\mathbf{D}}\,\overline{\mathbf{B}}^{-1}\overline{\mathbf{A}} & \overline{\mathbf{D}}\,\overline{\mathbf{B}}^{-1}\end{pmatrix}\begin{pmatrix}\overline{\mathbf{r}}_0\\\overline{\mathbf{r}}\end{pmatrix}\right]d^3\tilde{\mathbf{r}}_{10}d^3\tilde{\mathbf{r}}_{20}, \tag{10}$$

where

$$\overline{\mathbf{r}}_0=\begin{pmatrix}\tilde{\mathbf{r}}_{10}\\ \tilde{\mathbf{r}}_{20}\end{pmatrix}=\begin{pmatrix}x_{10}\\ y_{10}\\ \tau_{10}\\ x_{20}\\ y_{20}\\ \tau_{20}\end{pmatrix} \quad\text{and}\quad \overline{\mathbf{r}}=\begin{pmatrix}\tilde{\mathbf{r}}_{1}\\ \tilde{\mathbf{r}}_{2}\end{pmatrix}=\begin{pmatrix}x_{1}\\ y_{1}\\ \tau_{1}\\ x_{2}\\ y_{2}\\ \tau_{2}\end{pmatrix} \tag{11}$$

represent the initial two arbitrary spatial-temporal points and the output two arbitrary spatial-temporal points, respectively; and m is only related to the dimensions of the spatial-temporal point $\tilde{\mathbf{r}}$, now $m=3$. In Eq. (10), we define

$$\overline{\mathbf{A}}=\begin{pmatrix}\tilde{\mathbf{A}}_1 & 0\\ 0 & \tilde{A}_2\end{pmatrix},\quad \overline{\mathbf{B}}=\begin{pmatrix}\tilde{\mathbf{B}}_1 & 0\\ 0 & -\tilde{\mathbf{B}}_2\end{pmatrix},\quad \overline{\mathbf{C}}=\begin{pmatrix}\tilde{\mathbf{C}}_1 & 0\\ 0 & -\tilde{\mathbf{C}}_2\end{pmatrix},$$

$$\overline{\mathbf{D}}=\begin{pmatrix}\tilde{\mathbf{D}}_1 & 0\\ 0 & \tilde{\mathbf{D}}_2\end{pmatrix}. \tag{12}$$

Due to the scalar property of the exponential kernel in Eq. (10), we can get the following relations:

$$(\overline{\mathbf{B}}^{-1}\overline{\mathbf{A}})^T=\overline{\mathbf{B}}^{-1}\overline{\mathbf{A}},\quad (\overline{\mathbf{D}}\overline{\mathbf{B}}^{-1})^T=\overline{\mathbf{D}}\overline{\mathbf{B}}^{-1},$$
$$\overline{\mathbf{C}}-\overline{\mathbf{D}}\overline{\mathbf{B}}^{-1}\overline{\mathbf{A}}=-(\overline{\mathbf{B}}^{-1})^T. \tag{13}$$

Formula (10) describes the propagation of the correlation function between different spatiotemporal points of any pulsed beam through second-order dispersive media. Using this formula, we can easily get the evolution information about the correlation functions of partially coherent spatiotemporal pulsed beams.

III. DEFINITION OF PARTIALLY COHERENT GSMPB

As we know, in free space, the light field of a complete coherent pulsed Gaussian beam without spatiotemporal coupling has the following form:

$$U(x,y,z,t)=\exp\left[-\frac{x^2+y^2}{4\sigma_I^2}\right]\exp\left[-\frac{\tau^2}{2\sigma_\tau^2}\right]\exp[i\omega_0\tau], \tag{14}$$

where σ_I and σ_τ are the spot radius and pulse temporal width of the pulsed Gaussian beam, respectively; and $\tau=t-z/c$ is the relative delay time. Here c is the light speed in vacuum. The definition of the correlation function [1,2] reads

$$\Gamma(x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2)=\langle U^*(x_2,y_2,z_2,t_2)U(x_1,y_1,z_1,t_1)\rangle. \tag{15}$$

Substituting Eq. (14) into Eq. (15), the correlation function between any two spatial-temporal points (x_1,y_1,z_1,t_1) and (x_2,y_2,z_2,t_2) in the spatiotemporal domain can be written as

$$\Gamma(x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2)=\exp\left[-\frac{x_1^2+y_1^2}{4\sigma_I^2}\right]\exp\left[-\frac{x_2^2+y_2^2}{4\sigma_I^2}\right]\exp\left[-\frac{\tau_1^2}{2\sigma_\tau^2}\right]\times\exp\left[-\frac{\tau_2^2}{2\sigma_\tau^2}\right]\exp[i\omega_0(\tau_1-\tau_2)] \tag{16}$$

Also, by the definition of the complex degree of coherence [1,2],

$$\gamma(x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2)=\frac{\Gamma(x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2)}{[I(x_1,y_1,z_1,t_1)I(x_2,y_2,z_2,t_2)]^{1/2}}, \tag{17}$$

where $I(x_i,y_i,z_i,t_i)=\Gamma(x_i,y_i,z_i,t_i;x_i,y_i,z_i,t_i)$ $(i=1,2)$ represents the intensity of pulsed beam, we can get the complex degree of the coherence of such coherent pulsed beams

$$\gamma(x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2)=\exp[i\omega_0(\tau_1-\tau_2)], \tag{18}$$

and consequently $|\gamma(x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2)|=1$. This indicates that the conventional pulsed Gaussian beam is fully coherent between all spatiotemporal points.

In practice, any pulse is not perfectly coherent in both space and time. How do we describe this kind of spatiotemporal Gaussian pulsed beams? First, we consider the case in which the space and time variables of the pulsed beam are separable. We introduce two new parameters to characterize the spatial and temporal coherence of a pulse, and assume that both spatial coherence and the temporal coherence are in Gaussian. In such a case, there is no coupling between spatial coherence and temporal coherence, so we have the complex degree of the coherence as follows:

$$\gamma(x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2)=\exp\left[-\frac{(x_1-x_2)^2+(y_1-y_2)^2}{2\sigma_{cs}^2}\right]\exp\left[-\frac{(\tau_1-\tau_2)^2}{2\sigma_{ct}^2}\right]\times\exp[i\omega_0(\tau_1-\tau_2)], \tag{19}$$

where σ_{cs} may be called the transverse spatial coherence width, and σ_{ct} may be called the temporal coherence width or longitudinal correlation width. In analogy to the form of Gaussian Schell-model beams (GSMBs) [1,2], we may call such pulsed beams Gaussian Schell-model pulsed beams. From Eqs. (17) and (19), we know that the shape (the intensity as a function of space and time) of the pulsed beam is not changed, therefore we get the generalized correlation function of the pulsed beams with the help of the correlation function. The result is

$$\Gamma(x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2)=[I(x_1,y_1,z_1,t_1)I(x_2,y_2,z_2,t_2)]^{1/2}\gamma(x_1,y_2,z_1,t_1;x_2,y_2,z_2,t_2)$$
$$=\exp\left[-\frac{x_1^2+y_1^2}{4\sigma_I^2}\right]\exp\left[-\frac{x_2^2+y_2^2}{4\sigma_I^2}\right]\exp\left[-\frac{\tau_1^2}{2\sigma_\tau^2}\right]\exp\left[-\frac{\tau_2^2}{2\sigma_\tau^2}\right]\exp\left[-\frac{(x_1-x_2)^2+(y_1-y_2)^2}{2\sigma_{cs}^2}\right]$$
$$\times\exp\left[-\frac{(\tau_1-\tau_2)^2}{2\sigma_{ct}^2}\right]\exp[i\omega_0(\tau_1-\tau_2)]. \tag{20}$$

The above equation can be rearranged into a more compact form as follows:

$$\Gamma(\overline{\mathbf{r}})=\exp\left[-\frac{ik_0}{2}\overline{\mathbf{r}}^T\overline{\mathbf{Q}}^{-1}\overline{\mathbf{r}}\right]\exp[i\omega_0(\tau_1-\tau_2)], \tag{21}$$

where the superscript "T" stands for transpose:

$$\overline{\mathbf{r}}=\begin{pmatrix}\tilde{\mathbf{r}}_1\\ \tilde{\mathbf{r}}_2\end{pmatrix}\quad\text{and}\quad\tilde{\mathbf{r}}_i=\begin{pmatrix}x_i\\ y_i\\ \tau_i\end{pmatrix}\quad(i=1,2). \tag{22}$$

$\overline{\mathbf{Q}}^{-1}$ is a 6×6 matrix given by

$$\overline{\mathbf{Q}}^{-1}=\begin{pmatrix}-\frac{i}{2k_0}\boldsymbol{\sigma}_I^{-2}-\frac{i}{k_0}\boldsymbol{\sigma}_{cs}^{-2}, & \mathbf{0}_1, & \frac{i}{k_0}\boldsymbol{\sigma}_{cs}^{-2}, & \mathbf{0}_1\\ \mathbf{0}_2, & -\frac{i}{k_0}\sigma_\tau^{-2}-\frac{i}{k_0}\sigma_{ct}^{-2}, & \mathbf{0}_2, & \frac{i}{k_0}\sigma_{ct}^{-2}\\ \frac{i}{k_0}\boldsymbol{\sigma}_{cs}^{-2}, & \mathbf{0}_1 & -\frac{i}{2k_0}\boldsymbol{\sigma}_I^{-2}-\frac{i}{k_0}\boldsymbol{\sigma}_{cs}^{-2}, & \mathbf{0}_1\\ \mathbf{0}_2, & \frac{i}{k_0}\sigma_{ct}^{-2} & \mathbf{0}_2, & -\frac{i}{k_0}\sigma_\tau^{-2}-\frac{i}{k_0}\sigma_{ct}^{-2}\end{pmatrix}, \tag{23}$$

where

$$\boldsymbol{\sigma}_I^{-2}=\begin{pmatrix}\sigma_I^{-2} & 0\\ 0 & \sigma_I^{-2}\end{pmatrix},\quad \boldsymbol{\sigma}_{cs}^{-2}=\begin{pmatrix}\sigma_{cs}^{-2} & 0\\ 0 & \sigma_{cs}^{-2}\end{pmatrix}.$$

Here $\mathbf{0}_1$ is a 2×1 zero matrix and $\mathbf{0}_2$ is a 1×2 zero matrix, which indicate that there is no spatiotemporal intensity coupling and no spatiotemporal coherence coupling as we have assumed above. We may call $\overline{\mathbf{Q}}^{-1}$ the spatiotemporal complex pulsed beam parameter matrix. It can describe mainly properties of the pulsed beam. We can extract many quantities, such as the temporal width, transverse spot size, spatial coherence, and temporal coherence, etc., from this spatiotemporal complex pulsed beam parameter matrix.

For the more general cases, there always exist spatiotemporal coupling including intensity coupling and coherence coupling, e.g. when a pulsed beam passes through a nonideal dispersive optical element [14,15]. In order to get the analytic results, we neglect the initial transverse spatial wavefront curvature [16], the initial chirp coefficient [17], and the twisted factor [16,18]. Under such assumptions, $\overline{\mathbf{Q}}^{-1}$ becomes

$$\overline{\mathbf{Q}}^{-1}=\begin{pmatrix}-\frac{i}{2k_0}\boldsymbol{\sigma}_I^{-2}-\frac{i}{k_0}\boldsymbol{\sigma}_{cs}^{-2}, & \mathbf{q}_{12}, & \frac{i}{k_0}\boldsymbol{\sigma}_{cs}^{-2}, & \mathbf{q}_{14}\\ \mathbf{q}_{21}, & \frac{-i}{k_0}\sigma_\tau^{-2}-\frac{i}{k_0}\sigma_{ct}^{-2}, & \mathbf{q}_{23}, & \frac{i}{k_0}\sigma_{ct}^{-2}\\ \frac{i}{k_0}\boldsymbol{\sigma}_{cs}^{-2}, & \mathbf{q}_{32} & -\frac{i}{2k_0}\boldsymbol{\sigma}_I^{-2}-\frac{i}{k_0}\boldsymbol{\sigma}_{cs}^{-2}, & \mathbf{q}_{34}\\ \mathbf{q}_{41}, & \frac{i}{k_0}\sigma_{ct}^{-2} & \mathbf{q}_{43}, & -\frac{i}{k_0}\sigma_\tau^{-2}-\frac{i}{k_0}\sigma_{ct}^{-2}\end{pmatrix}, \tag{24}$$

where $\boldsymbol{\sigma}_I^{-2}$ and $\boldsymbol{\sigma}_{cs}^{-2}$ are all 2×2 matrices with transpose symmetry, given by

$$\boldsymbol{\sigma}_I^{-2}=\begin{pmatrix}\sigma_{Ixx}^{-2} & \sigma_{Ixy}^{-2}\\ \sigma_{Ixy}^{-2} & \sigma_{Iyy}^{-2}\end{pmatrix},\quad \boldsymbol{\sigma}_{cs}^{-2}=\begin{pmatrix}\sigma_{csxx}^{-2} & \sigma_{csxy}^{-2}\\ \sigma_{csxy}^{-2} & \sigma_{csyy}^{-2}\end{pmatrix}. \tag{25}$$

Where σ_{Ixx} and σ_{Iyy} are the transverse spot of the pulsed beam in x and y directions, respectively; σ_{Ixy} is the intensity coupling between x and y variables. σ_{csxx}^{-2} and σ_{csyy}^{-2} are the transverse coherences in the x and y directions, respectively; σ_{csxy}^{-2} is the coherence coupling between x and y variables. Here the parameters $\mathbf{q}_{12}$, $\mathbf{q}_{32}$, $\mathbf{q}_{14}$, and $\mathbf{q}_{34}$ are 2×1 nonzero real matrices, and $\mathbf{q}_{21}$, $\mathbf{q}_{23}$, $\mathbf{q}_{41}$, and $\mathbf{q}_{43}$ are 1×2 nonzero real matrices, whose elements depend on the coupling between space and time in the pulsed beams. At the same time, by the characteristics of the correlation function $\Gamma^*(\tilde{r}_1;\tilde{r}_2)=\Gamma(\tilde{r}_2;\tilde{r}_1)$, there exist the relations $\mathbf{q}_{12}=\mathbf{q}_{21}^T=\mathbf{q}_{34}=\mathbf{q}_{43}^T$ and $\mathbf{q}_{14}=\mathbf{q}_{41}^T=\mathbf{q}_{32}=\mathbf{q}_{23}^T$, and we may call the parameters $\mathbf{q}_{12}$, $\mathbf{q}_{21}$, $\mathbf{q}_{34}$, and $\mathbf{q}_{43}$ the spatiotemporal intensity *self-coupling* at the same spatiotemporal point and the parameters $\mathbf{q}_{14}$, $\mathbf{q}_{41}$, $\mathbf{q}_{23}$, and $\mathbf{q}_{32}$ the spatiotemporal correlation mutual-coupling between any two different spatiotemporal points. We may call both the two couplings as the spatial-temporal coupling in a general case. Equations (21), (24), and (25) describe the generalized partially coherent Gaussian Schell-model pulses. We will discuss the propagation behavior of such pulsed beams in the following.

IV. *ABCD* LAW OF PARTIALLY COHERENT GSMPBs

Now we consider the propagation of partially coherent GSMPBs through dispersive media. Substituting Eq. (21) into Eq. (10), we can obtain

$$\Gamma(\overline{\mathbf{r}})=\{\det[(\overline{\mathbf{A}}+\overline{\mathbf{B}}\overline{\mathbf{Q}}_i^{-1})]\}^{-1/2}\exp\left[-\frac{ik_0}{2}\overline{\mathbf{r}}^T\overline{\mathbf{Q}}_o^{-1}\overline{\mathbf{r}}\right], \tag{26}$$

where $\overline{\mathbf{Q}}_i^{-1}$ and $\overline{\mathbf{Q}}_o^{-1}$ denote the input and output partially coherent spatial-temporal complex curvature tensor, respectively. They are related by the following relation:

$$\overline{\mathbf{Q}}_o^{-1}=(\overline{\mathbf{C}}+\overline{\mathbf{D}}\overline{\mathbf{Q}}_i^{-1})(\overline{\mathbf{A}}+\overline{\mathbf{B}}\overline{\mathbf{Q}}_i^{-1})^{-1}. \tag{27}$$

During the calculations, we omitted the factor $\exp[i\omega_0(\tau_1-\tau_2)]$ in Eq. (21) due to the slowly varying condition of Eq. (1). It should be pointed out that Eqs. (26) and (27) describe the transformation of the general GSMPBs propagating in dispersive media, and can be used to deal with many related problems such as the compression and broadening of pulse widths, and spatial spreading of pulses and spatiotemporal coupling. Formula (27) is the *nonstationary* generalized ABCD law for nonstationary GSMPBs compared with the cases for statistically stationary partially coherent beams [16,19,20]. All previous results are only fitted to the statistically stationary fields for partially coherent light beams [19,20] or to the fully temporal correlated for coherent Gaussian pulses [14,15,17]. Now our formula connects these two limits and can deal with the effects of the transverse spatial coherence and the coherence time at the same time. Of course this formula can be reduced to the previous results for partially coherent light such as GSMBs and anisotropic GSMBs if we let the duration and temporal coherence time width of partially coherent GSMPBs be infinity.

In the following, we will give an example to illustrate the application of Eqs. (26) and (27). At the same time, we show the effects of the transverse spatial coherence on the spatiotemporal behavior of the pulsed beam due to the spatial-temporal coupling.

V. EXAMPLE

In many situations, we often consider the propagation from one plane to another perpendicular to the propagation axis; thus we have the relations $\xi_{10}=\xi_{20}=\xi_0$, $\xi_1=\xi_2=\xi$, $\tilde{\mathbf{A}}_1=\tilde{\mathbf{A}}_2=\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}_1=\tilde{\mathbf{B}}_2=\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}_1=\tilde{\mathbf{C}}_2=\tilde{\mathbf{C}}$, and $\tilde{\mathbf{D}}_1=\tilde{\mathbf{D}}_2=\tilde{\mathbf{D}}$. For simplicity, we only consider the two-dimension pulsed beam, i.e., only including one transverse direction such as x, and another longitudinal time scale τ. We also consider the simplest cases in which the pulsed beam propagates in free space. Therefore, the spatiotemporal matrices of the free space have the following simple forms:

$$\tilde{\mathbf{A}}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \tilde{\mathbf{B}}=\begin{pmatrix}B&0\\0&0\end{pmatrix},\quad \tilde{\mathbf{C}}=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad \mathbf{D}=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{28}$$

Using the above results in the previous sections, in Eq. (24) $\overline{Q}^{-1}$ is simplified into

$$\overline{Q}^{-1}=\begin{pmatrix}
-\frac{i}{2k_0}\sigma_I^{-2}-\frac{i}{k_0}\sigma_{cs}^{-2}, & q_{12}, & \frac{i}{k_0}\sigma_{cs}^{-2}, & q_{14}\\
q_{12}, & -\frac{i}{k_0}\sigma_\tau^{-2}-\frac{i}{k_0}\sigma_{ct}^{-2}, & q_{14}, & \frac{i}{k_0}\sigma_{ct}^{-2}\\
\frac{i}{k_0}\sigma_{cs}^{-2}, & q_{14} & -\frac{i}{2k_0}\sigma_I^{-2}-\frac{i}{k_0}\sigma_{cs}^{-2}, & q_{12}\\
q_{14}, & \frac{i}{k_0}\sigma_{ct}^{-2} & q_{12}, & \frac{-i}{k_0}\sigma_\tau^{-2}-\frac{i}{k_0}\sigma_{ct}^{-2}
\end{pmatrix}, \tag{29}$$

where all the elements of the matrix are scalar quantities.

First, we will consider that the initial partially coherent pulse has no spatiotemporal coupling, i.e., these parameters q_{12} and q_{14} are zero in Eq. (29). Then we can get the explicit output expression of such a partially coherent pulse through free space as follows:

$$\Gamma(x_1,\tau_1,x_2,\tau_2)=G(B)\exp\left\{-\frac{k_0^2}{4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})}[(2\sigma_{cs}^{-2}+\sigma_I^{-2})(x_1^2+x_2^2)-4\sigma_{cs}^{-2}x_1x_2]\right\}$$
$$\times\exp\left\{-\frac{1}{2}[\sigma_{ct}^{-2}(\tau_1-\tau_2)^2+\sigma_\tau^{-2}(\tau_1^2+\tau_1^2)]\right\}\exp\left\{-\frac{iBk_0\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})(x_1^2-x_2^2)}{2[4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})]}\right\}, \tag{30}$$

where

$$G(B)=\frac{1}{\left(1+\frac{B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})}{4k_0^2}\right)^{1/2}}.$$

From the above formula, we can find that for the pulsed beam without the initial spatial-temporal coupling, the behaviors of the spatial and temporal evolutions are independent of each other. Meanwhile, the transverse spatial evolution of such a pulse is similar to that of a partially coherent Gaussian Schell-model beam in a spatial domain [2]; the temporal evolution is invariant in such special cases.

Next, we assume that there exist spatiotemporal coupling initially, i.e., these parameters q_{12} and q_{14} are nonzero in Eq. (29). We can also obtain the following expression for such partially coherent spatial-temporal correlation coupled pulsed beams:

$$\Gamma(x_1,\tau_1,x_2,\tau_2)=G(B)\exp\left\{-\frac{k_0^2}{4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})}[(2\sigma_{cs}^{-2}+\sigma_I^{-2})(x_1^2+x_2^2)-4\sigma_{cs}^{-2}x_1x_2]\right\}$$
$$\times\exp\left\{-\frac{1}{2}[\sigma_{ct}^{-2}(\tau_1-\tau_2)^2+\sigma_\tau^{-2}(\tau_1^2+\tau_1^2)]\right.$$
$$\left.-\frac{2k_0^2B^2\sigma_{cs}^{-2}(q_{12}+q_{14})^2(\tau_1+\tau_2)^2+k_0^2B^2\sigma_I^{-2}[(q_{12}\tau_1+q_{14}\tau_2)^2+(q_{14}\tau_1+q_{12}\tau_2)^2]}{4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})}\right\}$$
$$\times\exp\left\{-\frac{2k_0^2B\sigma_I^{-2}[q_{14}(\tau_1x_2-\tau_2x_1)+q_{12}(\tau_2x_2-\tau_1x_1)]+4k_0^2B\sigma_{cs}^{-2}(q_{12}+q_{14})(x_2-x_1)(\tau_1-\tau_2)}{4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})}\right\}$$
$$\times\exp\left\{-\frac{ik_0B\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})(x_1^2-x_2^2)}{2[4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})]}\right\}\exp\left\{\frac{i2k_0^3B(q_{12}^2-q_{14}^2)(\tau_2^2-\tau_1^2)}{4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})}\right\}$$
$$\times\exp\left\{-\frac{i4k_0^3[q_{12}(\tau_1x_1+\tau_2x_2)+q_{14}(\tau_1x_2+\tau_2x_1)]}{4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})}\right\}. \tag{31}$$

From Eq. (31), we can obtain the changes of the effective transverse spatial radius and the effective temporal width of the pulse in free space:

$$\sigma_I^{(\mathrm{eff})}=\left(\frac{4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})}{2k_0^2\sigma_I^{-2}}\right)^{1/2}, \tag{32}$$

$$\sigma_\tau^{(\mathrm{eff})}=\frac{1}{\left(\frac{1}{\sigma_{\mathrm{addition}}^2}+\frac{1}{\sigma_\tau^2}\right)^{1/2}}, \tag{33}$$

where

$$\sigma_{\mathrm{addition}}^2=\frac{4k_0^2+B^2\sigma_I^{-2}(4\sigma_{cs}^{-2}+\sigma_I^{-2})}{2k_0^2B^2(4\sigma_{cs}^{-2}+\sigma_I^{-2})(q_{12}+q_{14})^2}. \tag{34}$$

Compared with Eq. (30), we find that, the spatial behavior of the coupling pulse is still similar to pulse without spatiotemporal coupling; but the temporal behavior becomes much more complicated due to the effects of spatiotemporal coupling. Figure 1 shows the differences of the evolution of the temporal width under the different transverse spatial coher-

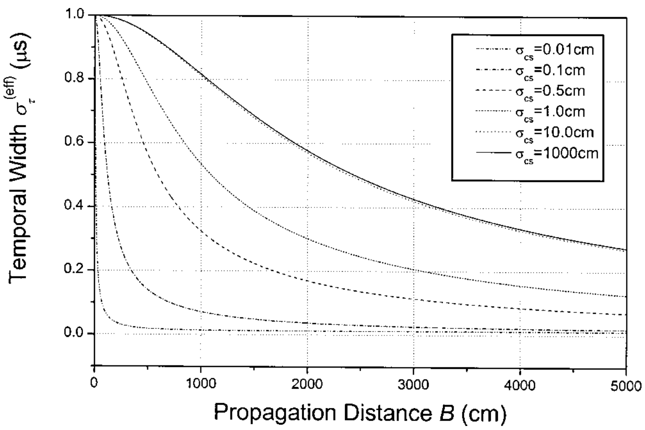

FIG. 1. The evolution of the temporal width under different transverse spatial coherences $\sigma_{\rm cs}$ for pulsed beams with a spatiotemporal coupling. $k_0 = 59275.3\ \mathrm{cm}^{-1}$ (corresponding to $\lambda_0 = 1.06\ \mu\mathrm{m}$), $\sigma_\tau = 1.0\ \mu\mathrm{s}$, $\sigma_I = 1.0$ cm, and $q_{12} = q_{14} = 0.5 \times 10^{-3}\ (\mu\mathrm{s}^{-1})$.

ence σ_{cs} for pulsed beams with spatiotemporal coupling. In fact, it has been known that, due to the spatiotemporal coupling, the change of the temporal width of the pulse will become very complicated [14,15]. Now in our cases, we show the effects of the transverse spatial coherence on the temporal width of the pulsed beam due to spatiotemporal coupling, even in free space. If the media are dispersive materials or other optical elements such as a dispersive prism and a pair of gratings, the behavior of the temporal width of the pulse will be more complicated. But, using Eq. (27), we can easily get the related information.

VI. CONCLUSIONS

In this paper, we discussed the propagation of nonstationary partially coherent pulsed beams. The propagation formulas for any nonstationary pulsed beams in dispersive media are derived according to the basic concepts of coherence theory in a spatiotemporal domain. Then we have introduced a general model to describe partially coherent Gaussian Schell-model pulsed beams (GSMPBs), and derived the propagation formula for the GSMPBs rigorously through dispersive media and obtained the nonstationary generalized *ABCD* law to describe the spatiotemporal behavior of the partially coherent GSMPBs. This transformation law can deal with the problems of partially coherent pulses passing through the complicated optical systems more conveniently and powerfully. A simple application example showing the effects of spatial coherence on the evolutions of the temporal width of pulsed beam in free space is given.

ACKNOWLEDGMENTS

This work was supported by FRG from Hong Kong Baptist University and National Natural Science Foundation of China (60078003).

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